Some Coherent Excitations for the (2+1)-Dimensional Generalized Broer-Kaup System

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Z. Naturforsch. **61a,** 116 – 124 (2006); received February 13, 2006

Using a projective Riccati equation, several types of solutions of the (2+1)-dimensional generalized Broer-Kaup system are obtained, including multiple soliton solutions, periodic soliton solutions and Weierstrass function solutions. From these, two sets of wave packets are expressed as rational functions of elliptic functions. Especially, peculiar wave patterns that are localized in one direction but periodic in the other direction arise by taking the long wave length limit in one spatial variable. Also exponentially localized wave patterns, which differ from the known dromions, are obtained by taking the long wave length limit in both spatial variables. The interactions of two dromions with inelastic and elastic behaviors are presented.

Key words: Generalized Broer-Kaup System; Projective Riccati Equation; Coherent Excitation; Dromion.

1. Introduction

The dynamics of localized structures is a fascinating and important subject in nonlinear science. Many exotic one-dimensional localized excitations, such as kinks, breathers, instantons, and peakons, have been studied earlier in the literature. Similar studies in higher spatial dimensions are less numerous. Discoveries and understanding of (2+1)-dimensional (2 spatial and 1 temporal) nonlinear evolution equations (NEEs) have seen progress at a tremendous pace in recent years [1 – 4].

In one spatial dimension a periodic wave can usually be regarded as a superposition of an infinite array of equally spaced identical solitons [5]. The corresponding situation for NEEs in higher spatial dimensions is much less well understood. Though some types of doubly periodic solutions have been obtained [6]. Their analysis has either not been performed or not been completed yet. In some special cases these doubly periodic patterns can be regarded as the generalization of a two-solitoff, a singly periodic perturbed line soliton or a single straight line kink soliton [7]. Since the dromion is the fundamental coherent structure in (2+1)-dimensions, it would be natural to investigate if doubly periodic wave patterns can be regarded as a two-dimensional superposition of arrays of dromions.

The (2+1)-dimensional generalized Broer-Kaup (GBK) system is chosen as an illustrative example here for several reasons. First, the dromion has been studied intensively for this system. Second, a special procedure, named here as the mapping approach [8] via a projective Riccati equation [9, 10], is established and will lead to exact solutions for the (2+1)-dimensional GBK system. By choosing elementary functions as the building blocks in this algorithm, various localized solutions can be found. In the present approach, the classical Jacobi elliptic functions will be employed as the building blocks, resulting in doubly periodic wave patterns for the GBK system.

The celebrated (2+1)-dimensional GBK system [11] for the three functions u, v, w is defined by

$$u_t - u_{xx} + 2uu_x + w_x + Ew + Fv = 0, \tag{1}$$

$$\begin{aligned} &v_t + 2(uv)_x + v_{xx} + 4E(v_x - u_{xy}) \\ &\quad + 4F(v_y - u_{yy}) + G(v - 2u_y) = 0, \end{aligned} \tag{2}$$

$$w_y - v_x = 0, \tag{3}$$

where E, F, G are arbitrary constants, and it was recently derived from a typical (1+1)-dimensional Broer-Kaup (BK) system [12] by means of the Painlevé analysis [11]. Obviously, when $E = F = G = 0$, the

GBK system degenerates to the celebrated (2+1)-dimensional BK system [13], which can be derived from an inner parameter dependent symmetry constraint of the Kadomtsev-Petviashvili model [14]. Using some suitable dependent and independent variable transformations, Chen and Li [15] have proven that the (2+1)-dimensional BK system can be transformed to the (2+1)-dimensional dispersive long wave equation (DLWE) [16] and the (2+1)-dimensional Ablowitz-Kaup-Newell-Segur (AKNS) system [17]. Actually, the (2+1)-dimensional BK system has been widely investigated by many researchers [18]. However, to the best of our knowledge, its excitations obtained here with the aid of a projective Riccati equation approach have not been reported in the preceding literature though Zheng et al. [19] derived a variable separation solution to the GBK system by a special Painlevé-Bäcklund transformation.

It is shown in Section 2 that the novel exact solutions, including multiple soliton solutions, periodic soliton solutions and Weierstrass function solutions for the GBK system, are derived by a special mapping transformation procedure. Section 3 deals with the nonlinear coherent structures of the GBK system. Doubly periodic, semi-localized and localized structures are investigated. The last section consists of a short summary and discussion.

2. Novel Solutions of the (2+1)-Dimensional GBK System

Letting $f \equiv f(\xi(X)), g \equiv g(\xi(X))$ [where $\xi \equiv \xi(X)$ is a still undetermined function of the independent variables $X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$], the projective Riccati equation [9, 10] is defined by

$$f' = pfg, \quad g' = q + pg^2 - rf, \tag{4}$$

where $p^2 = 1$, q and r are two real constants. When $p = -1$ and $q = 1$, (4) reduces to the coupled equations given in [9]. The following relation between f and g can be satisfied:

$$g^2 = -\frac{1}{p}[q - 2rf + \frac{r^2+\delta}{q} f^2], \tag{5}$$

where $\delta = \pm 1$. Equation (4) has ever been discussed in [10]. In this paper, we discuss some other cases.

Lemma. If the condition of (5) holds with other choices of δ, the projective Riccati equation (4) has the following solutions:

(a) If $\delta = -r^2$, the Weierstrass elliptic function solution is admitted:

$$f = \frac{q}{6r} + \frac{2}{pr}\wp(\xi), \quad g = \frac{12\wp'(\xi)}{q + 12p\wp(\xi)}. \tag{6}$$

Here $p = \pm 1$, the Weierstrass elliptic function $\wp(\xi) = \wp(\xi; g_2, g_3)$ satisfies $\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3$, and $g_2 = \frac{q^2}{12}$, $g_3 = \frac{pq^3}{216}$.

(b) If $\delta = -\frac{r^2}{25}$, the projective Riccati equation has the Weierstrass elliptic function solution

$$f = \frac{5q}{6r} + \frac{5pq^2}{72r\wp(\xi)}, g = -\frac{q\wp'(\xi)}{\wp(\xi)(12\wp(\xi) + pq)}, \tag{7}$$

where $p = \pm 1$. Both q and r in (6) and (7) are arbitrary constants.

(c) If $\delta = h^2 - s^2$ and $pq < 0$, (4) has the solitary solution

$$\begin{aligned} f &= \frac{q}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \\ g &= -\frac{\sqrt{-pq}}{p}\frac{s\sinh(\sqrt{-pq}\xi) + h\cosh(\sqrt{-pq}\xi)}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \end{aligned} \tag{8}$$

where $p = \pm 1$, s and h are arbitrary constants.

(d) If $\delta = -h^2 - s^2$ and $pq > 0$, we have the trigonometric function solution

$$\begin{aligned} f &= \frac{q}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \\ g &= \frac{\sqrt{pq}}{p}\frac{s\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \end{aligned} \tag{9}$$

where $p = \pm 1$, s and h are arbitrary constants.

(e) If $q = 0$, (4) has the rational solution

$$\begin{aligned} f &= \frac{2}{pr\xi^2 + C_1\xi - C_2}, \\ g &= -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}, \end{aligned} \tag{10}$$

where C_1, C_2, and r are arbitrary constants, and $p = \pm 1$.

We now introduce the mapping approach via the above projective Riccati equation. The basic idea of the algorithm is: Considering a nonlinear partial differential equation (NPDE) with independent variables $X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$, and the dependent variable $u \equiv u(X)$,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{11}$$

where P is a polynomial function of its arguments and the subscripts denote the partial derivatives, we assume that its solution is written as the standard truncated Painlevé expansion, namely

$$u = A_0(X) + \sum_{i=1}^{n} (A_i(X) f(\xi(X)) + B_i(X) g(\xi(X))) f^{i-1}(\xi(X)). \tag{12}$$

Here $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 1, \ldots, n)$ are arbitrary functions to be determined, and f, g satisfy the projective Riccati equation (4).

To determine u explicitly, one proceeds as follows: First, similar to the usual mapping approach, we can determine n by balancing the highest-order nonlinear term with the highest-order partial derivative term in (11). Second, substituting (12) with (4) and (5) into the given NPDE, collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, \cdots, j = 0, 1)$ and eliminating each of them, we can derive a set of partial differential equations for $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 1, ..., n)$ and $\xi(X)$. Third, to calculate $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 1, ..., n)$ and $\xi(X)$, we solve these partial differential equations. Finally, substituting $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 1, ..., n)$, $\xi(X)$, and the solutions (6)–(10) into (12), one obtains solutions of the given NPDE.

Now, we apply the above mapping approach to the GBK system. We first differentiate (1) with respect to the variable y once and substitute (3) into it. The GBK system is then changed into a set of coupled nonlinear partial differential equations:

$$(u_t - u_{xx} + 2uu_x)_y + v_{xx} + Ev_x + Fv_y = 0, \tag{13}$$

$$v_t + 2(uv)_x + v_{xx} + 4E(v_x - u_{xy}) + 4F(v_y - u_{yy}) + G(v - 2u_y) = 0. \tag{14}$$

According to the balancing procedure, (12) becomes

$$u = a_0 + a_1 f(\xi) + b_1 g(\xi),$$
$$v = A_0 + A_1 f(\xi) + A_2 f^2(\xi) + B_1 g(\xi) + B_2 f(\xi) g(\xi), \tag{15}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 and ξ are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting (15) with (4) and (5) into (13) and (14), collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, 2, 3, 4,\ j = 0, 1)$ and setting each of the coefficients to zero, we can derive a set of partial differential equations for a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 and ξ. It is difficult to obtain the general solution of these algebraic equations based on the solutions of (4). Fortunately, in the special case that $\xi = \chi(x,t) + \varphi(y)$, where $\chi(x,t)$, $\varphi(y)$ are two arbitrary variable separated functions of (x,t) and y, respectively, we can obtain solutions of (13) and (14).

Theorem. For the (2+1)-dimensional GBK system (1)–(3), there are five couples of variable separated solutions, related to the projective Riccati equation (4).

(a) For $\delta = -r^2$, the Weierstrass elliptic function solution is:

$$u = -\frac{2E\chi_x + \chi_{xx} + \chi_t + 2F\varphi_y}{2\chi_x} - \frac{1}{2} p\chi_x g(\xi),$$
$$v = -pr\chi_x\varphi_y f(\xi), \tag{16}$$

where $p = \pm 1$, and f, g are expressed by (6).

(b) For $F = 0$ and $\delta = -\frac{r^2}{25}$, another set of Weierstrass elliptic function solutions is found:

$$u = -\frac{2E\chi_x + \chi_{xx} + \chi_t}{2\chi_x} \pm \frac{r}{5}\sqrt{-\frac{6p}{q}}\chi_x f(\xi) - \frac{1}{2} p\chi_x g(\xi),$$
$$v = -pr\chi_x\varphi_y f(\xi) + \frac{24pr^2\chi_x\varphi_y f^2(\xi)}{25q} \pm \frac{2pr}{5}\sqrt{-\frac{6p}{q}}\chi_x\varphi_y f(\xi) g(\xi), \tag{17}$$

where $p = \pm 1$, q and r in (16) are arbitrary constants, f and g are expressed by (7).

(c) For $F = 0$, $\delta = h^2 - s^2$, and $pq < 0$, a couple of solitary solutions is:

$$u = -\frac{2E\chi_x + \chi_{xx} + \chi_t}{2\chi_x} \pm \frac{1}{2}\sqrt{-\frac{p(h^2 + r^2 - s^2)}{q}}\chi_x f(\xi) - \frac{1}{2} p\chi_x g(\xi),$$
$$v = -pr\chi_x\varphi_y f(\xi) + \frac{p(h^2 + r^2 - s^2)\chi_x\varphi_y f^2(\xi)}{q} \pm p\sqrt{-\frac{p(h^2 + r^2 - s^2)}{q}}\chi_x\varphi_y f(\xi) g(\xi), \tag{18}$$

where $p = \pm 1$, s and h are arbitrary constants, f and g are expressed by (8).

(d) For $F = 0$, $\delta = -h^2 - s^2$, and $pq > 0$, the trigonometric function solutions are:

$$\begin{aligned} u &= -\frac{2E\chi_x + \chi_{xx} + \chi_t}{2\chi_x} \\ &\quad \pm \frac{1}{2}\sqrt{\frac{p(h^2 - r^2 + s^2)}{q}}\chi_x f(\xi) - \frac{1}{2}p\chi_x g(\xi), \\ v &= -pr\chi_x\varphi_y f(\xi) - \frac{p(h^2 - r^2 + s^2)\chi_x\varphi_y f^2(\xi)}{q} \\ &\quad \pm p\sqrt{\frac{p(h^2 - r^2 + s^2)}{q}}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \tag{19}$$

where $p = \pm 1$, s and h are arbitrary constants, f and g are expressed by (9).

(e) For $F = 0$ and $q = 0$, a pair of rational solutions is:

$$\begin{aligned} u &= -\frac{2E\chi_x + \chi_{xx} + \chi_t}{2\chi_x} \\ &\quad \pm p(\sqrt{C_1^2 + 4C_2 pr} f(\xi) + 2g(\xi))\chi_x, \\ v &= -pr\chi_x\varphi_y f(\xi) - p^2(\frac{1}{4}C_1^2 - C_2 pr)\chi_x\varphi_y f^2(\xi) \\ &\quad \mp \frac{p^2}{2}\sqrt{C_1^2 + 4C_2 pr}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \tag{20}$$

where C_1, C_2, and r are arbitrary constants, $p = \pm 1$, f and g are expressed by (10).

3. Nonlinear Coherent Structures of the (2+1)-Dimensional GBK System

3.1. Case A

It is known that for a nonlinear system, by choosing elliptic functions with different, independent moduli for $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$, doubly periodic patterns can be obtained. For instance, if we take $p = -1$, $q = 5$, $r = 3$, $s = 2$ and $h = 1$, then the absolute value V of the function v in (18) henceforth denoted as "physical quantity" becomes

$$V = \left| \frac{5(3 + (3 - 2\sqrt{6})\sinh(\sqrt{5}(\chi + \varphi)) + (6 - \sqrt{6})\cosh(\sqrt{5}(\chi + \varphi)))}{(3 + \sinh(\sqrt{5}(\chi + \varphi)) + 2\cosh(\sqrt{5}(\chi + \varphi)))^2}\chi_x\varphi_y \right|. \tag{21}$$

If the two arbitrary functions χ and φ are chosen as

$$\chi = \alpha^{-1}\arcsin(\mathrm{sn}(\alpha(x+t), n_1)), \quad \varphi = \beta^{-1}\arcsin(\mathrm{sn}(\beta y, n_2)), \tag{22}$$

where α, β are arbitrary constants, n_1, n_2 are moduli of the Jacobi elliptic functions which satisfy $\chi_x = \mathrm{dn}(\alpha(x+t), n_1)$, $\varphi_y = \mathrm{dn}(\beta y, n_2)$, the physical quantity V defined in (21) shows a special type of doubly periodic pattern. Figure 1a displays V expressed by (21) with the condition (22), where the parameters are chosen as $\alpha = \beta = 1$, $n_1 = n_2 = 0.1$ and the time $t = 0$. However, by taking one of the moduli to be unity, patterns periodic in one direction but localized in the other are obtained. Figure 1b illustrates one of these scenarios when the modulus n_1 is allowed to tend to 1 (for this limit, the Jacobi elliptic function sn degenerates into the hyperbolic tanh function and is no longer periodic), but the other parameters are the same as those in Figure 1a. The remarkable case occurs when both moduli n_1, n_2 are allowed to tend to 1, but the constants $\alpha = \beta = 1$, and the time still is $t = 0$, the formulas (21) and (22) yield an exponentially localized structure (Fig. 1c):

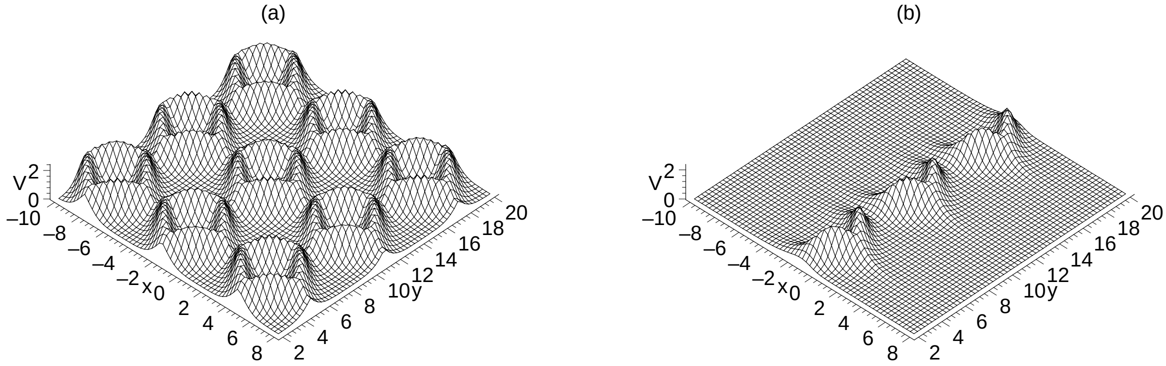
$$V_1 = \sqrt{1 - \tanh^2 x}\sqrt{1 - \tanh^2 y}\frac{|15 - (10\sqrt{6} - 15)\sinh(\sqrt{5}\Delta_1) - (5\sqrt{6} - 30)\cosh(\sqrt{5}\Delta_1)|}{(3 + \sinh(\sqrt{5}\Delta_1) + 2\cosh(\sqrt{5}\Delta_1))^2}, \tag{23}$$

where

$$\Delta_1 = \arcsin(\tanh x) + \arcsin(\tanh y).$$

One can then conclude that this expression is a dromion, defined here loosely as an exponentially localized solution. However, (23) is different from the known, conventional dromion [2,4], which is typically given entirely in terms of exponential functions only.

It is worth to note that for other values of t these graphs are shifted along the x-axis, because χ' depends on $x + t$ only.



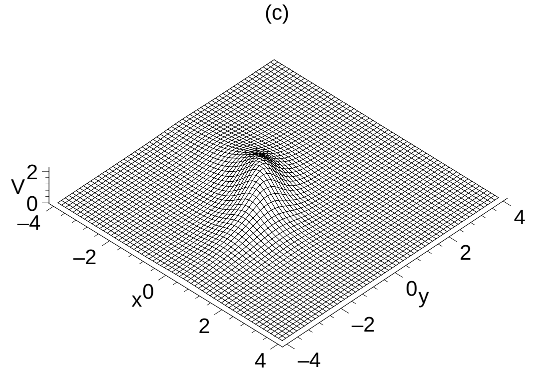


Fig. 1. (a) Structure of the physical quantity V expressed by (21) with the condition (22), the parameters $\alpha = \beta = 1$, $n_1 = n_2 = 0.1$ and the time $t = 0$. (b) Semi-localized structure of the physical quantity V expressed by (21) for the condition (22) but the parameter n, now chosen as $n_1 = 1$, with the other constants the same as those in (a). (c) A dromion structure of the physical quantity V_1 expressed by (23). The structures in (a) and (b) are continuing periodically in both x and y [for (a)] or in y only [for (b)].

3.2. Case B

Second, when choosing χ and φ as

$$\begin{aligned}\chi &= (3\alpha)^{-1}[\arcsin(\mathrm{sn}(\alpha(x+t),n_1))]^3,\\ \varphi &= (3\beta)^{-1}[\arcsin(\mathrm{sn}(\beta y,n_2))]^3,\end{aligned} \tag{24}$$

so that $\chi_x = [\arcsin(\mathrm{sn}(\alpha(x+t),n_1))]^2\mathrm{dn}(\alpha(x+t),n_1)$, $\varphi_y = [\arcsin(\mathrm{sn}(\beta y,n_2))]^2\mathrm{dn}(\beta y,n_2)$, we can derive a rather novel periodic pattern for the physical quantity V in (21). Figure 2a shows the structure of this novel solution V_2, which is given by:

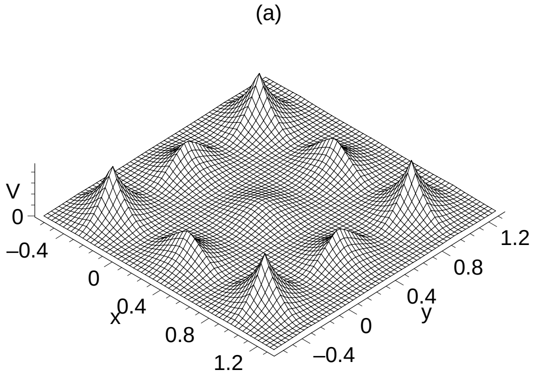
$$\begin{aligned}V_2 = &\frac{|\arcsin(\mathrm{sn}^2(5x,0.1))\mathrm{cn}(5x,0.1)\mathrm{dn}(5x,0.1)\arcsin(\mathrm{sn}^2(5y,0.1))\mathrm{cn}(5y,0.1)\mathrm{dn}(5y,0.1)|}{\sqrt{1-\mathrm{sn}^2(5x,0.1)}\sqrt{1-\mathrm{sn}^2(5y,0.1)}}\\ &\cdot\frac{|15-(10\sqrt{6}-15)\sinh\Delta_2-(5\sqrt{6}-30)\cosh\Delta_2|}{(3+\sinh\Delta_2+2\cosh\Delta_2)^2},\end{aligned} \tag{25}$$

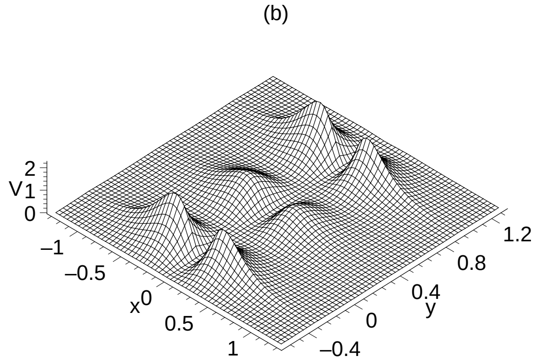
where

$$\Delta_2 = \frac{\sqrt{5}}{15}(\arcsin(\mathrm{sn}^3(5x,0.1))+\arcsin(\mathrm{sn}^3(5y,0.1)))$$

and cn and dn are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind. Here, we choose the parameters $\alpha = \beta = 5$, $n_1 = n_2 = 0.1$ and without loss of generallity the time $t = 0$. For a cross section in y, there are two peaks with different amplitudes in one period, and the same is true for a cross section in x.

Similarly, the long wave length limits of (24) are also instructive. Here, for $n_1 = 1$, $n_2 = 0.1$ the pattern is localized in the x direction but periodic in y, with four peaks per period (Fig. 2b), two higher and two smaller ones, each for $x > 0$ and $x < 0$, respectively. With n_1 and n_2 tending to one simultaneously, a novel dromion with four non-identical peaks separated by two perpendicular narrow gaps appears (Fig. 2c). This is not a 4-dromion structure, as a conventional 4-dromion has four underlying path lines, and the four dromions are located at the intersections of these lines. The dromion here (Fig. 2c) is driven by two perpendicular lines.





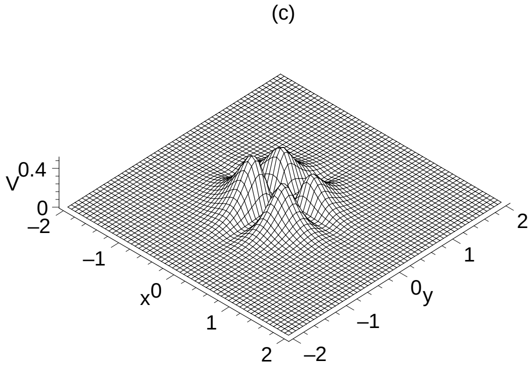


Fig. 2. (a) The doubly periodic structure of the physical quantity V_2 expressed by (25). (b) Semi-localized structure of the physical quantity V expressed by (21) for the condition (24); the parameters are chosen as $\alpha = \beta = 5$, $n_1 = 1$, $n_2 = 0.1$ and the time $t = 0$. (c) Localized structure of the physical quantity V expressed by (21) for the condition (24) and the parameters $n_1 = n_2 = 1$, other constants are the same as those in (b). Since x depends on $x+t$, the same structure arise for $t \neq 0$ but shifted along the x-axis.

3.3. Case C

The interaction of dromions deserves further investigations. The interaction can be elastic or inelastic. It is called elastic, if the amplitude, velocity and wave shape of two solitons do not change after their interaction. Fusion or fission of component solitons has also been observed to interact in an elastic or an inelastic way. In order to study the interaction property of the new dromion solution shown in Fig. 2c, we first write down the expression for a 2-dromion solution in the original coordinates:

$$\chi = \frac{\arcsin(\tanh(\alpha_1(x+v_1t)))^3}{3\alpha_1} + \frac{\arcsin(\tanh(\alpha_2(x+v_2t)))^3}{3\alpha_2}, \tag{26}$$

$$\varphi = \frac{\arcsin(\tanh(\beta y))^3}{3\beta}. \tag{27}$$

We set one of the two dromions to move along the negative x-axis with the velocity $v_1 = 5$ and the other is static, i. e., $v_2 = 0$. Figure 3 shows the collision of these two dromions. Initially $(t = -0.5)$, the dromion located at the point (0,0) is stationary and the other one, located at the point (2.5,0) is moving towards the stationary one (Figs. 3a and b). At $t = 0$, they merge to form a single entity (Fig. 3c) and then separate again (Figs. 3d and e). Eventually, at time $t = 0.5$, the moving one reaches the point $(-2.5,0)$ and the resting one is still at (0,0). We can regard this interaction process as one where the two dromions totally exchange their shapes with their velocities preserved. Alternatively, we can interpret the interaction such that the initially moving soliton has come to rest after the collision and the momentum has been totally transferred to the dromion initially at rest, but moving at the later time. Either way, the interaction between these two dromions is inelastic.

However, when taking

$$\chi = \frac{\arcsin(\tanh(\alpha_1(x+v_1t)))^2}{2\alpha_1} + \frac{\arcsin(\tanh(\alpha_2(x-v_2t)))^4}{4\alpha_2}, \tag{28}$$

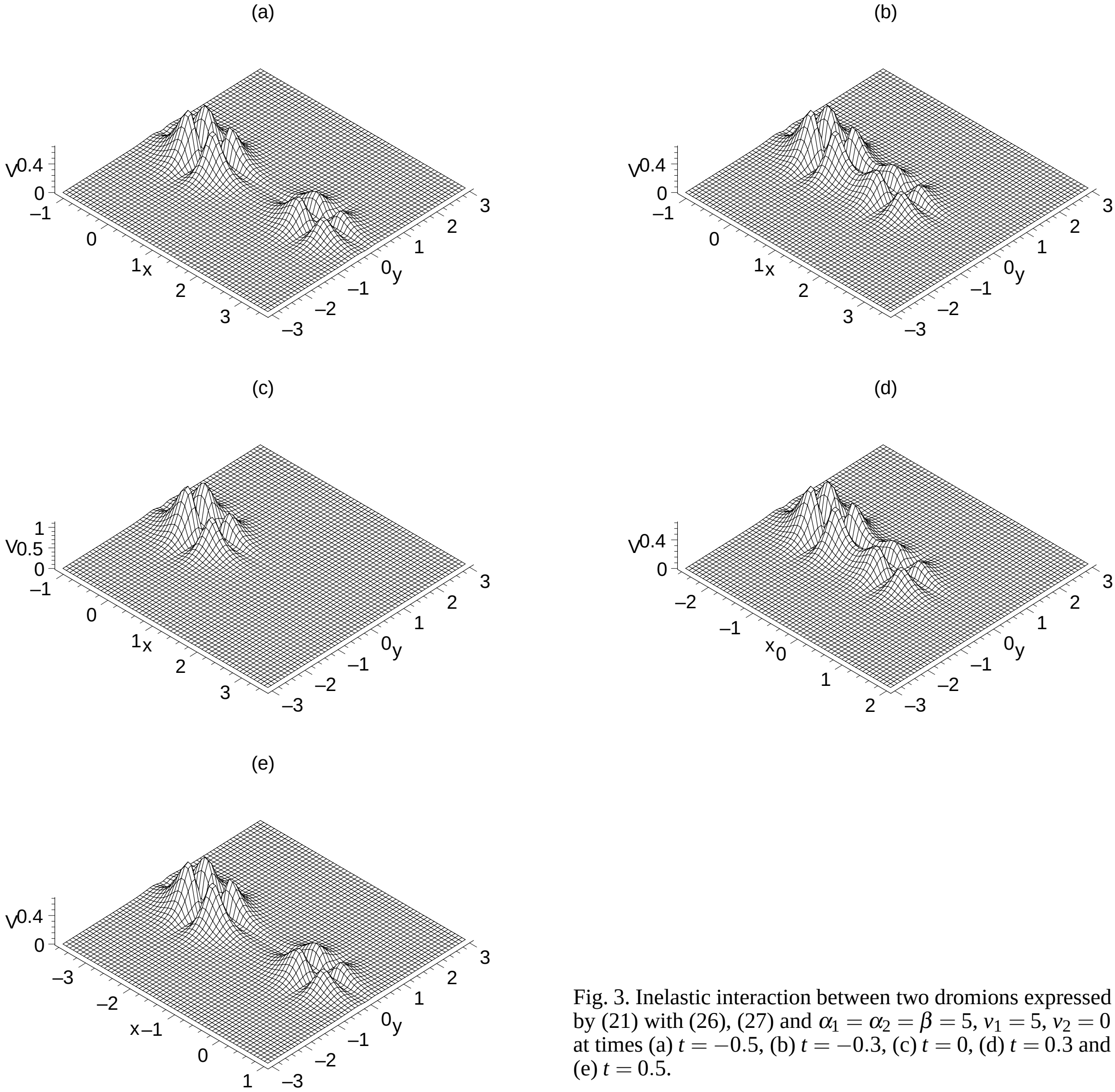


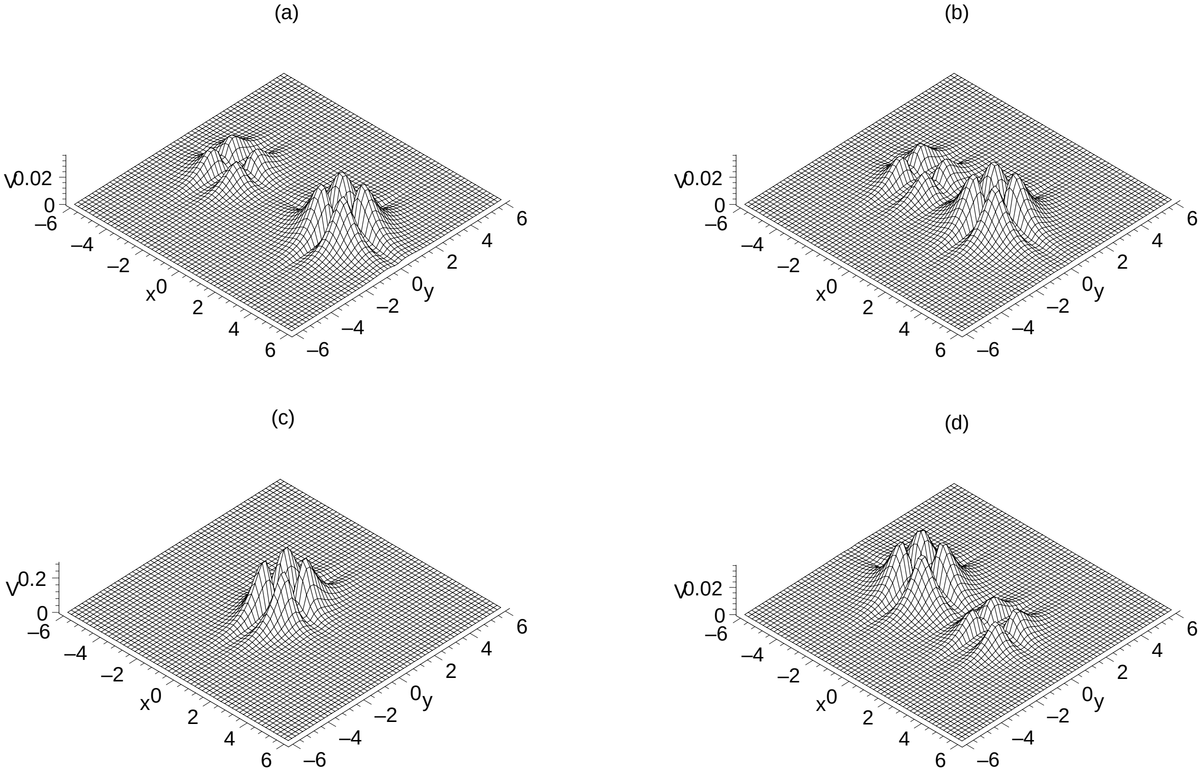
Fig. 3. Inelastic interaction between two dromions expressed by (21) with (26), (27) and $\alpha_1 = \alpha_2 = \beta = 5$, $v_1 = 5$, $v_2 = 0$ at times (a) $t = -0.5$, (b) $t = -0.3$, (c) $t = 0$, (d) $t = 0.3$ and (e) $t = 0.5$.

$$\varphi = \frac{\arcsin(\tanh(\beta y))^2}{2\beta}, \tag{29}$$

which means the interaction of two dromions moving along the x-axis, but in the opposite directions, we observe that the physical quantity V in (21) shows elastic behavior. Figure 4 shows an evolutional profile of the corresponding physical quantity V. From Fig. 4 and through detailed analysis, we find that the shape, amplitude and velocity of the two dromions are completely preserved after their interaction.

4. Summary and Discussion

In summary, with the use of the projective Riccati equation, we have obtained several types of exact solutions for the (2+1)-dimensional generalized Broer-Kaup (GBK) system, including multiple soliton solutions, periodic soliton solutions and Weierstrass function solutions. Some doubly periodic structures of the GBK system have been obtained and studied by employing the classical Jacobi elliptic functions as the building blocks for the variable separated solution. For



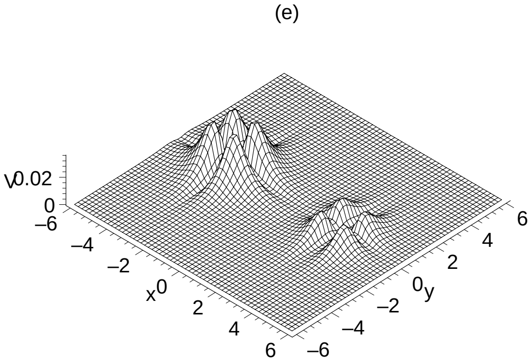


Fig. 4. Elastic interaction between two dromions expressed by (21) with (28), (29) and $\alpha_1 = \beta = 1$, $\alpha_2 = 2$, $v_1 = v_2 = 1$ at times (a) $t = -3$, (b) $t = -2$, (c) $t = 0$, (d) $t = 2$ and (e) $t = 3$.

minimal algebraic complexity the attention is restricted to cases where analytical, closed form expressions are obtained. The internal structures of the wave packet, in terms of the number of local maxima and minima, depend on the choice of elliptic functions and also on the two distinct, independent moduli. A semi-localized pattern, which is periodic in one direction, but localized in the other, can be obtained by choosing one of the moduli to be unity.

New exponentially localized units, or dromions, result if both moduli are tending to 1 as a limit. They are different from the conventional expressions and we believe that they deserve further study. In fact some of the new localized units discussed in this paper have four distinct peaks, but they are not a 4-dromion, because the underlying structures of the wave packets are different. The interaction between two dromions of this type is studied. One interpretation shows that they preserve their velocities, but totally exchange their shapes without phase shifts during the interaction. While the other interpretation shows that the shape, amplitude and velocity of the two dromions are completely preserved after their interaction.

A final remark concerning the constant q. The present choice of q ($q = 5$) is only for convenience, and the other choices of q will only lead to the modification of the detailed shape of the solution, but without changes of the properties of the solution.

Because of the wide application and the complexity of the doubly periodic structures related to the Jacobi elliptic functions, more details about these types of exact solutions deserve further study.

Acknowledgements

The authors would like to express their sincere thanks to the referee for his valuable suggestions and kind help. The work was supported by the National Natural Science Foundation of China (No. 10272071), the Natural Science Foundation of Zhejiang Province of China (No. Y604106), and the Scientific Research Fund of Zhejiang Provincial Education Department of China (No. 20051356).

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